THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VII. APRIL, 1900. No. 4.

A POPULAR ACCOUNT OF SOME NEW FIELDS OF THOUGHT IN MATHEMATICS.

By DR. G. A. MILLER.

Read at the regular Winter Term Meeting of the Alpha Chapter of Sigma Xi, Cornell University.

At the beginning of the nineteenth century elementary arithmetic was a Freshman subject in our best colleges. In 1802 the standard of admission to Harvard College was raised so as to include a knowledge of arithmetic to the 'Rule of Three.' A boy could enter the oldest college in America prior to 1803 without a knowledge of a multiplication table.* From that time on the entrance requirements in mathematics were rapidly increased, but it was not until after the founding of Johns Hopkins University that the spirit of mathematical investigation took deep root in this country.

The lectures of Sylvester and Cayley at Johns Hopkins University, the founding of the *American Journal of Mathematics* and the young men who received their training abroad coöperated to spread the spirit of mathematical investigation throughout our land. This has led to the formation of the American Mathematical Society eight years ago as well as to the starting of a new research journal, *The Transactions of the American Mathematical Society*, at the beginning of this year. While these were some of the results of mathematical activity, they, in a still stronger sense, tend to augment this activity.

In Europe such men as Descartes, Newton, Leibniz, Lagrange and Euler

*Cajori, *The History and Teaching of Mathematics*, 1890, page 60.

laid the foundation for the development of mathematics in many directions. These men, as well as a few of the most prominent names in the early part of the nineteenth century, were not specialists in mathematics. They were familiar with all the fields of mathematical activity in their day and some of them were well known for their contributions in other fields of knowledge. The last three-quarters of a century and especially the last two or three decades have witnessed a marvelous change in the mathematical activity of Europe. Mathematical periodicals have sprung up on all sides. A number of mathematical societies have been organized and many of the leading mathematicians have confined their investigations to comparatively small fields of mathematics.

The rapid increase of the mathematical literature created an imperative need of bibliographical reviews. This need was met in part by the establishment at Berlin, in 1869, of a year-book devoted exclusively to the reviews of mathematical articles, *Jahrbuch über die Fortschritte der Mathematik.* The 28th volume of this work reached our library a short time ago. It contains over 900 pages, and gives a review of over 2000 memoirs and books. With a view towards further increasing the facilities to keep in touch with the growing literature, the Amsterdam Mathematical Society commenced the publication of a semi-annual review, *Revue Semestrielle*, in 1893. In the last number of this, 236 periodicals are quoted, each of which contains, at times, mathematical articles that are of sufficient merit to be noted. Each of the four countries, France, Germany, Italy, and America publishes over thirty such periodicals.

One of the characterestic features of our times is the prominence of the spirit of coöperation. The mathematical periodicals and the mathematical societies are evidences of this spirit. In quite recent years international mathematical congresses have given further expression of the wide-spread desire to coöperate with even the most remote workers in the same fields. The first of these congresses was held in Zurich in 1897, and the second is to be held during the coming summer in connection with the Paris Exposition. The same spirit led in 1894 to the starting of a periodical, *L'intérmediaire des Mathématiciens*, which is devoted exclusively to the publishing of queries and answers in regard to different mathematical subjects.

This desire for extensive coöperation is tending towards unifying mathematics and towards laying especial stress on those subjects which have the widest application in the different mathematical disciplines. This explains why the theory of functions of a complex variable and the theory of groups are occupying such prominent places in recent mathematical thought.*

Before entering upon a description of some of the fields included in these subjects and the interesting problems which they present, it may be well to state explicitly that our remarks on mathematics will have very little reference to its application to other sciences. To the pure mathematician a result that has extensive application in mathematics is just as important and useful as one which applies to the other sciences. Mathematics is a science which deserves to be de-

*Cf. Klein, *Chicago Mathematical Papers*, 1893, page 134.

veloped for its own sake. The thought that some of its results may find application in other sciences is, however, a continual inspiration, and those who investigate such applications sometimes add materially to the development of mathematics.

The curve representing a function of a single variable was the principal object of investigation during the eighteenth and a great part of the nineteenth century.* The investigations of Abel and Cauchy on power series during the early part of the nineteenth century furnished the foundation for the modern theory of analytic functions—a theory which has been adorned by the labors of some of the most brilliant mathematicians of the preceding generation and which is claiming the attention of some of the foremost thinkers of the present time. Quite recently this theory has been made more accessible to English readers by the treatises of Forsyth and Harkness and Morley.

The critical spirit of our age is, in a large measure, due to the study of the theory of analytic functions. "Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivative."† When it was discovered that such functions exist and that the works of some of the greatest mathematicians of the preceding centuries had to be modified in some instances, mathematicians naturally became much more exacting in regard to rigor, and thus ushered in an age which may be compared with the times of Euclid with respect to its demands for rigor. Whether our critical age will produce a work which, like Euclid, will serve as a model for millenniums cannot be foretold, but it seems certain that works which can stand the critical tests of this age will stand the tests of all ages.

The critical spirit of our times is the foundation of what has been styled the *arithmetization of mathematics*. This movement which the late Weierstrass knew so well to lead is pervading more and more the whole mathematical world. We are rapidly banishing from our treatises the term quantity and replacing it by the word number. Our geometric intuitions are forced into the background and logical deductions from intuitions are taking their places. Who can conceive of curves which have no tangent at any of their rational points in a given interval? Nevertheless it is well known that such curves exist. An account of such functions was first published by Hankel in 1870.‡

Mathematicians find themselves in a great dilemma at this point. Geometric intuition has been such a strong instrument of research and has given so much life and beauty to mathematical investigation that mathematicians cling to it as to their own lives. It is an enormous price when rigor can be purchased only with geometric intuition. Yet, in the present stage of mathematical thought, this seems to be the only thing that will be accepted, and mathematicians stand helpless before this decree.

A few examples may throw some light on this subject. What do we un-

---

*Lie, *Leipziger Berichte*, Vol. 47, page 261.

†Klein, *Eveston Colloquium*, 1894, page 41.

‡Cf. Pierpont, *Bulletin of the American Mathematical Society*, Vol. 5, page 398.

derstand by the length of a continuous curve? The intuitionalist says, if we connect different points of the continuous curve by straight lines and find the sum of the lengths of these straight lines, then the length of the curve will be the limit of this sum as the number of the points is indefinitely increased. Jordan was the first to call attention to the fact that this sum need not have a limit. Hence there are continuous curves which do not have any length according to the ordinary definition of length. In fact a number of area-filling curves have recently been studied, and Cantor has shown that a multiplicity of any number of dimensions can be put in a one to one correspondence with a multiplicity of one dimension.

These are some of the facts which have compelled mathematicians to construct their own worlds—the number worlds. Conclusions drawn in one number world do not necessarily apply to another. When a problem is under consideration the number world is so chosen as to meet the demands of the problem. For instance, the constructions and demonstrations of Euclid's geometry seem to require only a space composed of quadratic numbers.* Hence it appears that we do not need to assume that space is continuous in order to demonstrate the theorems of elementary geometry. Similarly in algebra, we are laying more and more stress upon a distinct statement of the number world (the domain of rationality) in which we are operating. Such specifications add a clearness and rigor to our work which would otherwise scarcely be possible.

This refinement which the mathematical thought of to-day is so actively cultivating is not restricted to the finite region. Mathematical infinity is receiving its share of attention. It is well known that it is sometimes desirable to regard the infinite region as a single point. This is, for instance, the case in the transformation known as inversion. Again, in ordinary projective geometry, it is generally convenient to regard the infinite region as of one lower dimension than the finite, so that the infinite region of a plane is merely a line and the infinite region of space is a plane. The student of differential calculus is, moreover, familiar with the infinite variable and the many simplifications which its use makes possible.

The most fruitful investigations along this line are those on multiplicities (Mengenlehre, ensembles). Any total of definite and clearly defined elements is said to form a multiplicity. If two multiplicities are simply isomorphic, *i. e.*, if there is a 1, 1 correspondence between the elements of the multiplicites, they are said to be equivalent, or to have the same power. For example, it is easy to prove that all the positive rational numbers are equivalent to the natural numbers. To do this we may associate all the rational numbers p/q for which the sum $p+q=n$, any positive integer. We thus have the $n-1$ numbers.

$$\frac{n-1}{1},\quad \frac{n-2}{2},\quad \frac{n-3}{3}\cdots\cdots,\quad \frac{2}{n-2},\quad \frac{1}{n-1}.$$

We may let 1 correspond to 1; the numbers for which $n=3$ correspond to

*Cf. Strong, *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 443.

2 and 3; the numbers for which $n=4$ correspond to 4, 5 and 6, etc. We thus obtain the following 1, 1 corresponding between all the rational numbers and the positive integers:

$$\frac{1}{1};\ \frac{2}{1},\ \frac{1}{2};\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3};\ \frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4};\ \frac{5}{1},\ \frac{4}{2},\ \frac{3}{3},\ \frac{2}{4},\ \frac{1}{5};\ \ldots.$$
$$1;\ 2,\ 3;\ 4,\ 5,\ 6;\ 7,\ 8,\ 9,\ 10;\ 11,\ 12,\ 13,\ 14,\ 15;\ \ldots$$

It may be observed that we do not need to reduce the rational fractions to their lowest terms to effect this correspondence. This method of proof is due to Cantor, who has also proved that all algebraic numbers are equivalent to the natural numbers.* How important and far-reaching the investigations along this line are may be inferred from the fact that Jordan has employed them to serve as a foundation of the elementary part of the second edition of his magistral 'Cours d'analyse.'

A large number of mathematical problems may be reduced to equations involving a single unknown. The solution of such equations has occupied a prominent place in the mathematical literature for centuries. The problem is so difficult that it has been attacked from a number of different points and by means of a large variety of instruments. The instrument which has proved to be the most powerful and far-reaching is substitution groups. By means of it Abel succeeded in 1826 to prove that an equation whose degree exceeds four cannot generally be solved by the successive extraction of roots† and Galois a few years later sketched a far-reaching theory of equations which rests upon the theory of these groups.

In recent years it has been recognized (especially through the labors of Sophus Lie) that the theory of groups has very extensive and fundamental application in a large number of the other domains of mathematics. About a year ago the great French mathematician H. Poincaré, showed in an article, published in the Chicago *Monist*,‡ how this concept may be employed in laying the foundations of elementary geometry. It should be observed that the theory of groups is intrinsically based upon the fundamental concepts of mathematics. It is not a superstructure. It stands on its own foundation and supports more or less a number of other mathematical structures.

As this theory is less known than most of the other extensive branches of mathematics it may be desirable to enter into some details. It is evident that there are rational functions of n independent variables $(x_1, x_2, x_3, \ldots\ldots, x_n)$ which are not changed when these variables are permuted in every possible manner. Such functions are said to be symmetric in regard to these variables. The sum of any given power of these variables $(x_1{}^a+x_2{}^a+x_3{}^a+\ldots\ldots x_n{}^a)$ is an instance of a symmetric function. These functions occupy one extreme. The other extreme is furnished by functions such as $x_1+2x_2+3x_3+\ldots\ldots nx_n$ which change their value for every possible interchange of the variables. Most of the functions are of neither of these extreme types.

---

*Cantor, *Crelle*, Vol. 77, page 258; cf. Vol. 84, page 250.
†*Crelle*, Vol. 1.
‡October, 1898.

Suppose that a function $\phi(x_1, x_2, \ldots\ldots, x_n)$ is not changed by either of two interchanges of its independent variables. Such interchanges are called substitutions and they may be represented by S_1 and S_2. Since ϕ is not changed by either of the substitutions S_1, S_2, it cannot be changed by the substitutions which are equivalent to the succession of these substitutions taken in any order. All such substitutions may be represented by $S_1{}^{\alpha} S_2{}^{\beta} S_1{}^{\gamma} S_2{}^{\delta} \ldots\ldots$* Since only a finite number of permutations are possible with n letters it follows that $S_1{}^{\alpha} S_2{}^{\beta} S_1{}^{\gamma} S_2{}^{\delta} \ldots\ldots$ can represent only a finite number of distinct substitutions. The totality of these substitutions is said to be a *substitution group*. Hence we observe that every rational function belongs to some substitution group.

It was soon observed that an infinite number of functions belong to the same substitution group and that all of these functions can be expressed rationally in terms of one of them. The researches of Abel, Galois, and Jordan were based upon these facts and they show that the most important problems in the theory of equations involve the theory of substitution groups. The theory of groups was thus founded with a view to its application to a subject of paramount importance. A broad mathematical subject can, however, not grow vigorously and harmoniously as long as it is studied with a view to its direct applications to other mathematical subjects. It must be free to expand in all directions. That freedom for which the human race has ever been struggling must be vouchsafed to such fundamental subjects before they will exhibit their great fertility and far reaching connections. Less than three years ago the first work on the theory of groups that does not consider its application† was given to the public, but the mathematical journals have been publishing a large number of memoirs along the same line for a number of years.

In defining a substitution group we implied only two conditions; viz, no two substitutions of the group are identical and if we combine the substitutions in any way we obtain only substitutions which are already in the group. Substitutions obey *per se* some other conditions; *i. e.*, when they are combined (multiplied together) they obey the associative law and if we multiply a substitution by (or into) two different substitutions the products will be different. In general we say that any finite number of operations which obey these four conditions constitute a group; *e. g.*, all the rotations around the center of a regular solid which make the solid coincide with itself constitute a group, the n nth roots of unity constitute a group with respect to multiplication but not with respect to addition, etc.

While the bulk of the mathematicians are reveling in the new fields of thought which are opening up on all sides, without any thought in reference to any immediate practical application of their results, there is fortunately a goodly number whose main efforts are devoted towards making some of these new results useful to the investigators in some of the other sciences. As an instance

---

*The exponent indicates the number of times the substitution is employed in succession.
†Burnside, *Theory of Groups of a Finite Order*, 1897.

of fairly recent work of the latter kind, we may mention the study of linkages with a view towards describing a straight line. Although the straight line is of such fundamental importance both in pure and applied mathematics, yet it seems it was not until the latter half of the nineteenth century that a mechanical device has been discovered by means of which such a line can be described.

In 1864 M. Peaucellier, an officer of engineers in the French army, discovered the well known device to describe a straight line by means of an instrument composed of seven links. "His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian named Lipkin, who got a substantial reward from the Russian government for his supposed originality. However, M. Peaucellier's merit was finally recognized and he has been awarded the great mechanical prize of the Institute of France, the Prix Montyon."*

Although the straight line and the circle occupy such a prominent place in elementary geometry and have been before the eyes of the mathematicians for thousands of years, yet less than half a century has passed since the invention of a mechanical device by means of which the straight line can be drawn. Such discoveries go far towards emphasizing the need of investigations even in the most elementary subjects. Such investigations should, however, be preceded by a thorough knowledge of what has been done along the same lines.

If elementary mathematics is to continue to furnish the best possible preparation for the study of advanced mathematics, it is evident that it has to adapt itself to the rapid changes which are going on in the different branches of mathematics. A need is thus created for elementary text-books which meet the new requirements, and we are happy to be able to state that such books are being produced in our midst. How radical such changes may become cannot be foretold. In his address before the New York Mathematical Society, Simon Newcomb said, "The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution groups, as he might now but for inherited habits."† It is to be hoped that our inherited habits will not furnish an insurmountable barrier to progress in this direction.

In modern times the continent of Europe has always been the most progressive and most of the new theories were first developed in these countries. The theory of invariants seems to be an exception to this rule. The two great English mathematicians, Cayley and Sylvester, developed this theory with great vigor; when their important results became generally known on the continent (largely through the work of Clebsch), they aroused a great deal of interest and they furnished a starting point for many important investigations.

One of the fundamental processes of mathematics is transformation—the deducing of truths from given facts and relations. The expressions which remain invariant when given transformations are performed are naturally objects

---

*A. B. Kempe, *How to Draw a Straight Line*, page 12.

†*Bulletin of the New York Mathematical Society*, 1894, page 95.

of great interest and of fundamental importance. Imbued with this thought Lie once said, "What do the natural phenomena present to us if it is not a succession of infinitesimal transformations of which the laws of the universe are the invariants?"

It need scarcely be added that all mathematical thought even on the same subject is not running in the same channel. Klein has divided mathematicians into three main categories,* viz, the logicians, the formalists, and the intuitionists. The term logician is "intended to indicate that the main strength of the men belonging to this class lies in their logical and critical powers, in their ability to give strict definitions and to derive rigid deductions therefrom. The great and wholesome influence exerted in Germany by Weierstrass in this direction is well known. The formalists among the mathematicians excel mainly in the skillful formal treatment of a given question, in devising for it an algorithm. Gordan, or let us say Cayley or Sylvester, must be ranged in this group. To the intuitionists, finally, belong those who lay particular stress on geometrical intuition, not in pure geometry only, but in all branches of mathematics. What Benjamin Peirce has called 'geometrizing a mathematical question' seems to express the same idea. Lord Kelvin and von Staudt may be mentioned as types of this category."

In his address before the Zurich International Congress Poincaré says,† "Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has a philosophic end, and, I venture to say an esthetic end. It ought to incite the philosopher to search into the notions of number, space, and time; and above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of numbers and of forms; they are amazed when a new discovery discloses for them an unlooked-for perspective; and the joy they experience has it not the esthetic character although the senses take no part in it? Only the privileged few are called to enjoy it fully, it is true, but is it not the same with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake and that the theories not admitting of application to physics deserve to be studied as well as others. Moreover, a science produced with a view single to its applications is impossible; truths are fruitful only if they are concatenated; if we cleave to those only of which we expect immediate results the connecting links will be lacking, and there will be no longer a chain."

In closing we may remark that no effort has been made to mention all the new fields of mathematical thought. Mathematics, like the other sciences, seems to offer inexhaustable fields of investigation. As it expands its perimeter increases and hence there is a continually increasing demand for investigators. The fields that have been examined present many difficulties which cannot at present be surmounted. Some of the old difficulties are being removed by the light of the new discoveries. Still we know only a few things even about the

---

**The Evanston Colloquium*, page 2.

†*Revue Generale des Sciences*, Vol. 8, page 857.

fields which have been investigated. It is the exception that something can be done by known methods, the rule is that it cannot yet be done.

When we study the literature of some of the older subjects we are sometimes surprised by the large number of known facts, but when we come to study the subjects themselves and ask independent questions we are generally surprised to learn that so few properties are known. Hence it seems very desirable that the advanced student, at least, should study subjects rather than the known facts in regard to these subjects. In this way a more accurate idea of the true state of knowledge can be obtained. Besides the knowledge of having discovered facts and relations which will enter into the structure of a growing science is the greatest source of pleasure that the student can obtain.

Cornell University, March 9, 1900.

AN ELEMENTARY DERIVATION OF THE SERIES FOR $\sin x$ and $\cos x$.

By H. C. WHITAKER, Ph. D., Philadelphia, Pa.

Assume $\sin x = x + a_3x^3 + a_5x^5 + \ldots\ldots$ (1),

$\cos x = 1 + a_2x^2 + a_4x^6 + \ldots\ldots$ (2).

Then $\sin y = y + a_3y^3 + a_5y^5 + \ldots\ldots$ (3),

$\cos y = 1 + a_2y^2 + a_4y^6 + \ldots\ldots$ (4).

Also $\sin(x+y) = x + y + a_3(x+y)^3 + a_5(x+y)^5 + \ldots\ldots$ (5),

$\cos(x+y) = 1 + a_2(x+y)^2 + a_4(x+y)^4 + \ldots\ldots$ (6).

But $\sin(x+y) = \sin x\cos y + \cos x\sin y$ $\ldots\ldots$ (7),

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ $\ldots\ldots$ (8).

Hence the coefficients a_2, a_3, $a_4 \ldots\ldots$ can be determined by equating (5) to (1)×(4)+(2)×(3), and by equating (6) to (2)×(4)−(1)×(3). In the expansion, in each case, pick out only the terms containing the first power of y. In (5) and (7), the terms are

$$y + 3a_3x^2y + 5a_5x^4y +$$
$$y + a_2x^2y + a_4x^4y +$$

In (6) and (8), the terms are

$$2a_2xy + 4a_4x^3y + 6a_6x^5y +$$
$$-xy - a_8x^3y - a_5x^5y +$$